Gibbs partitions and feature allocations with product-form exchangeable probability functions

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Outline

1. Random partitions and random feature allocations

2. Characterizations of Gibbs-type partitions and product-form feature allocations

3. Example: stable beta scaled processes SB-SP

Definitions

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Example ($n = 4, k = 3$)

$$\rho_4 = \{\{2, 4\}, \{1, 2\}, \{3\}\}$$

Exchangeable random partition and feature allocation

Definition (Exchangeable random partition)

A random partition $\mathcal{P}_n$ of $[n]$ is called *exchangeable* if for any permutation $\sigma : [n] \longrightarrow [n]$ and every partition $\{G_1, \ldots, G_k\}$ of $[n]$,

$$P(\mathcal{P}_n = \{\sigma(G_1), \ldots, \sigma(G_k)\}) = P(\mathcal{P}_n = \{G_1, \ldots, G_k\})$$

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EPPF and EFPF

EPPF for exchangeable random partition

A random partition $\mathcal{P}_n$ of $[n]$ is *exchangeable*
$\Longleftrightarrow$ there exists a symmetric function $p_n : \mathcal{C}_n \longrightarrow [0, 1]$ such that, for every partition $\{G_1, \ldots, G_k\}$ of $[n]$,

$$P(\mathcal{P}_n = \{G_1, \ldots, G_k\}) = p_n(|G_1|, \ldots, |G_k|)$$

The function p_n is the EPPF of $\mathcal{P}_n$.

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EFPF for exchangeable random feature allocation

There exists a symmetric function $\pi_n : \cup_{k=0}^{\infty}[n]^k \longrightarrow [0,1]$ such that, for every ordered feature allocation $(A_1, \ldots, A_k)$ of $[n]$,

$$P(\tilde{F}_n = (A_1, \ldots, A_k)) = \pi_n(|A_1|, \ldots, |A_k|)$$

The function π_n is the EFPF of F_n
$\Longrightarrow$ the random feature allocation F_n of $[n]$ is *exchangeable.*

Consistency

Consider $(\mathcal{P}_n)_n$ (respectively, $(F_n)_n$) a sequence of exchangeable random partitions (respectively, feature allocations).

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Definition (Consistent sequence)

$(\mathcal{P}_n)_n$ (respectively, $(F_n)_n$) is *consistent* if the distribution of $\mathcal{P}_n$ (respectively, F_n) may be obtained from the distribution of $\mathcal{P}_{n+1}$ (respectively, F_{n+1}) by marginalizing out the $n+1$ individual.

Examples of e. c. random partitions

Example (The two-parameter family (also known as Ewens-Pitman family))

A random partition is a two-parameter (α, θ) process, if its EPPF has the form

$$\frac{(\theta+\alpha)_{k-1\uparrow\alpha}}{(\theta+1)_{n-1\uparrow}} \prod_{l=1}^{k} (1-\alpha)_{\lambda_j - 1\uparrow}$$

where $(x)_{m\uparrow\beta} = \prod_{i=0}^{m-1} (x + i\beta)$ and $(x)_{0\uparrow\beta} = 1$.

Parameters (α, θ) are such that either:

- $\alpha \in [-\infty, 0)$ and $\theta = m|\alpha|, m = 1, \ldots, \infty$ with proper definition in the limiting cases;
- $\alpha \in [0, 1]$ and $\theta \geq -\alpha$.

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Special case: the Ewens family ("Dirichlet Process") when $\alpha = 0$.

Examples of e. c. random feature allocations with EFPFs

Example (The 3-parameter Indian buffet Process (3IBP))

A feature allocation is a 3-parameter (γ, α, θ) IBP, with $\gamma \geq 0$, $\alpha \in [0, 1]$, $\theta \in [-\alpha, \infty)$, if it admits EFPF of the form

$$\frac{1}{k!}\left(\frac{\gamma}{(\theta+1)_{n-1\uparrow}}\right)^{k} \exp\left(-\sum_{i=1}^{n} \gamma \frac{(\alpha+\theta)_{i-1\uparrow}}{(1+\theta)_{i-1\uparrow}}\right) \prod_{l=1}^{k} (1-\alpha)_{m_l-1\uparrow}(\theta+\alpha)_{n-m_l\uparrow}$$

where $(x)_{m\uparrow} = \prod_{i=0}^{m-1}(x+i)$ and $(x)_{0\uparrow} = 1$.

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Special cases:

- the 2-parameter (γ, θ) IBP: $\alpha = 0$
- the 1-parameter γ IBP: $\alpha = 0, \theta = 1$

Example (The Beta-Bernoulli Process)

A feature allocation is a Beta-Bernoulli with parameters (N, α, θ), with $N \in \mathbb{N}$, $\alpha \in (-\infty, 0)$, $\theta \in [-\alpha, \infty)$, if it admits EFPF of the form

$$\binom{N}{k} \left(\frac{-\alpha}{(\theta + \alpha)_{n\uparrow}} \right)^k \left(\frac{(\theta + \alpha)_{n\uparrow}}{(\theta)_{n\uparrow}} \right)^N \prod_{l=1}^{k} (1 - \alpha)_{m_l - 1\uparrow} (\theta + \alpha)_{n - m_l\uparrow}$$

Gibbs-type partitions

Definition (Gnedin and Pitman 2004)

An exchangeable random partition is said to be of Gibbs form if for some nonnegative weights $W = (W_j)$ and $V = (V_{n,k})$, its EPPF satisfies

$$p_n(\lambda_1, \ldots, \lambda_k) = V_{n,k} \prod_{l=1}^{k} W_{\lambda_l} \tag{1}$$

for all $1 \leq k \leq n$ and all compositions $(\lambda_1, \ldots, \lambda_k)$ of n.

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Example

The two-parameter family (Ewens-Pitman family) is a class of Gibbs-type partitions.

Feature allocations with product-form EFPFs

Definition (Battiston et al. 2018)

An exchangeable feature allocation is said to have product-form EFPFs if for an infinite array $V = (V_{n,k} : (n,k) \in \mathbb{N} \times \mathbb{N}_0)$ and two sequences $W = (W_j : j \in \mathbb{N})$ and $U = (U_j : j \in \mathbb{N}_0)$ of nonnegative weights, its EFPF satisfies

$$\pi_n(m_1, \ldots, m_k) = V_{n,k} \prod_{l=1}^{k} W_{m_l} U_{n-m_l} \tag{2}$$

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Example

The 3-parameter Indian buffet Process (as well as the 2IBP and the 1IBP) and the Beta-Bernoulli process are exchangeable feature allocations with product-form EFPFs

Characterizations of the weights

Gibbs-type partitions

$\Longleftrightarrow$ (Gnedin and Pitman 2004)

Feature allocations with product-form

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* $\alpha = -\infty, q \in (0, 1)$:

$$W_m = q^{m-1}, \quad U_m = (1-q)^m$$

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Gibbs partitions and product-form features as mixtures

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- $\alpha \in [-\infty, 0)$: mixture over m of $(\alpha, |\alpha| m)$-partitions

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- $\alpha \in (0,1)$: mixture over γ of 3 -parameter IBPs;
- $\alpha = 0$: mixture over γ of 2 -parameter IBPs;
- $\alpha \in [-\infty, 0)$: mixture over N of Beta-Bernoulli models with N features

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From Proposition 4 of Camerlenghi et al. 2021, the EFPF of the feature allocation induced by the SB-SP (σ, c, β), with $\sigma \in (0,1), c, \beta > 0$ is

$$\pi_n(m_1, \ldots, m_k) = \frac{1}{k!} \frac{\sigma^k \beta^{c+1} \Gamma(k+c+1)}{(\beta + \gamma_0^{(n)})^{k+c+1} \Gamma(c+1)} \prod_{l=1}^{k} \frac{\Gamma(m_l - \sigma)\Gamma(n - m_l + 1)}{\Gamma(n - \sigma + 1)}$$

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$\Longrightarrow$ it is a mixture over γ of 3-parameter IBP $(\gamma, \sigma, 1-\sigma)$ distributions.

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Example ($n = 4, k = 5$)

$$\rho_4 = \{\{2, 4\}, \{1, 2\}, \{3\}, \{1, 2, 3\}, \{2, 5\}\}$$

Examples of e. c. random feature allocations with EFPFs

Example (The Beta-Bernoulli Process)

A feature allocation is a Beta-Bernoulli with parameters (N, α, θ), with $N \in \mathbb{N}$, $\alpha \in (-\infty, 0)$, $\theta \in [-\alpha, \infty)$, if it admits EFPF of the form

$$\binom{N}{k} \left(\frac{-\alpha}{(\theta + \alpha)_{n\uparrow}} \right)^k \left(\frac{(\theta + \alpha)_{n\uparrow}}{(\theta)_{n\uparrow}} \right)^N \prod_{l=1}^{k} (1 - \alpha)_{m_l - 1\uparrow} (\theta + \alpha)_{n - m_l\uparrow}$$

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For $\alpha \to -\infty, -\dfrac{\alpha}{\theta} \to q \in (0, 1)$, the homogeneous Bernoulli model of parameters (N, q) is obtained. In the following refer to the homogeneous Bernoulli model of parameters (N, q) as the special Beta-Bernoulli model with $\alpha = -\infty$ and characterize it with $q \in (0, 1)$.

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The distribution of a consistent exchangeable random partition is of Gibbs form, with $W_1 = V_{1,1}$ *if and only if one of the following two cases holds:*

(i) W *can be uniquely written as* $W_j = (1-\alpha)_{j-1\uparrow}$ *for fixed* $\alpha \in (-\infty, 1)$*, and the elements of* V *satisfy the recursion (with* $V_{1,1} = 1$*)*

$$V_{n,k} = (n - \alpha k) V_{n+1,k} + V_{n+1,k+1}, \qquad 1 \le k \le n$$

(ii) W *can be uniquely written as* $W_j = 1$ *and* V *satisfies the recursion (with* $V_{1,1} = 1$*)*

$$V_{n,k} = k V_{n+1,k} + V_{n+1,k+1}, \qquad 1 \le k \le n$$

corresponding to the limiting case $\alpha \to -\infty$ *(indicate this case with* $\alpha = -\infty$*).*

Characterization of weights in product-form feature allocations

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The distribution of a consistent exchangeable feature allocation has product-form EFPF if and only if

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The distribution of a consistent exchangeable feature allocation has product-form EFPF if and only if one of the following two cases holds:

(i) *W and U can be uniquely written as $W_m = (1-\alpha)_{m-1\uparrow}$ and $U_m = (\theta+\alpha)_{m\uparrow}$, for fixed α, θ satisfying $\alpha \in (-\infty, 1)$ and $\theta \in (-\alpha, \infty)$, and the elements of V satisfy the recursion*

$$V_{n,k} = \sum_{j=0}^{\infty} \binom{k+j}{j} \left((\theta+\alpha)_{n\uparrow}\right)^j (\theta+n)^k V_{n+1,k+j}$$

(ii) *W and U can be uniquely written $W_m = q^{m-1}$ and $U_m = (1-q)^m$, for some $q \in (0,1)$, and V satisfies the recursion*

$$V_{n,k} = \sum_{j=0}^{\infty} \binom{k+j}{j} (1-q)^{nj} V_{n+1,k+j}$$

corresponding to the limiting case $\alpha \to -\infty$ and $-\dfrac{\alpha}{\theta} \to q$ (indicate this case with $\alpha = -\infty$ and characterize it with $q \in (0,1)$).

Analogies between the characterizations of the weight sequences

Gibbs-type partitions

- sequence of weights: W

Feature allocations with product-form

- sequences of weights: W and U

Analogies between the characterizations of the weight sequences

Gibbs-type partitions

- sequence of weights: W
- W is parametrized by the single parameter $\alpha \in [-\infty, 1)$

Feature allocations with product-form

- sequences of weights: W and U
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* $\alpha = -\infty$ is the limiting case:

$$W_m = q^{m-1}, \quad U_m = (1-q)^m$$

still parametrized by the single parameter $q \in (0, 1)$

Gibbs-type partitions as mixtures

Theorem (Theorem 12 of Gnedin and Pitman 2004)

Each Gibbs partition of fixed type $\alpha \in [-\infty, 1)$ *is a unique probability mixture of the extreme partitions. In particular,*

(i) for $\alpha \in [-\infty, 0)$, *mixtures over* m *of* $(\alpha, |\alpha|m)$*-partitions with* $m = 1, \ldots, \infty$;

(ii) for $\alpha = 0$, *mixtures over* θ *of Ewens* $(0, \theta)$*-partitions with* $\theta \in [0, \infty]$;

(iii) for $\alpha \in (0, 1)$, *mixtures over* s *of Poisson-Kingman* $(\alpha|s)$*-partitions with* $s \in [0, \infty]$, *where Poisson-Kingman* (α, s)*-partition refers to the partition derived from a Poisson-Kingman discrete distribution denoted in Pitman 2003 by* $PK(\rho_\alpha|t)$ *for* $t = s^{-\alpha}$.

Product-form feature allocations as mixtures

Theorem (Battiston et al. 2018)

Each feature allocation with product-form EFPF of fixed parameters (α, θ) (or q if $\alpha = -\infty$) is a unique probability mixture of the extreme feature allocations. In particular,

(i) for $\alpha \in [-\infty, 0)$, mixtures over N of Beta-Bernoulli models with N features;

(ii) for $\alpha \in (0, 1)$, mixtures over γ of 3 -parameter IBPs;

(iii) for $\alpha = 0$, mixtures over γ of 2 -parameter IBPs.

Analogies between the characterizations of the V-weights

Once fixed the parameters α for the Gibbs type partition and (α, θ) for the feature allocation with product-form EFPF, each of them can be written as

Gibbs-type partitions

- $\alpha \in (0, 1)$: mixture over s of Poisson-Kingman $(\alpha|s)$-partitions;

Feature allocations with product-form

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Feature allocations with product-form

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Once fixed the parameters α for the Gibbs type partition and (α, θ) for the feature allocation with product-form EFPF, each of them can be written as

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- $\alpha \in (0, 1)$: mixture over s of Poisson-Kingman $(\alpha|s)$-partitions;
- $\alpha = 0$: mixture over θ of the Ewens $(0, \theta)$-partitions;
- $\alpha \in [-\infty, 0)$: mixture over m of $(\alpha, |\alpha|m)$-partitions

Feature allocations with product-form

- $\alpha \in (0, 1)$: mixture over γ of 3 -parameter IBPs;
- $\alpha = 0$: mixture over γ of 2 -parameter IBPs;
- $\alpha \in [-\infty, 0)$: mixture over N of Beta-Bernoulli models with N features