Exercise C

Consider a model in which $\mathbb{E}[y_i] = \beta x_i$, $\mathrm{var}(y_i) = \sigma^2 x_i$, with $x_i > 0$. By definition:

a.
$$\hat{\beta}_{WLS} = \underset{\beta \in \mathbb{R}}{\arg\min} \sum_{i=1}^{m} \frac{1}{x_i}(y_i - \beta x_i)^2 = \underset{\beta \in \mathbb{R}}{\arg\min}\, \ell_{WLS}(\beta).$$

$$\ell'_{WLS}(\beta) = -\sum_{i=1}^{m} \frac{2}{\cancel{x_i}}(y_i - \beta x_i)\cancel{x_i} = -2\sum_{i=1}^{m}(y_i - x_i\beta)$$

$$\Longrightarrow \cancel{-2}\sum_{i=1}^{m} y_i = \cancel{-2}\beta\sum_{i=1}^{m} x_i$$

$$\Longrightarrow \boxed{\hat{\beta}_{WLS} = \frac{\bar{y}}{\bar{x}}.}$$

(This is indeed a maximum.. as it can be checked from the second derivative)

$$\mathbb{E}[\hat{\beta}_{WLS}] = \frac{\mathbb{E}[\bar{y}]}{\bar{x}} = \frac{\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}[\beta x_i]}{\bar{x}} = \beta\frac{\bar{x}}{\bar{x}} = \beta.$$

$$\mathrm{var}(\hat{\beta}_{WLS}) = \frac{1}{\bar{x}^2}\cdot \mathrm{var}\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right) = \underbrace{\frac{1}{m^2\bar{x}^2}\sum_{i=1}^{m}\mathrm{var}(y_i)}_{\text{independence of } y_i.} = \frac{\sigma^2\sum_{i=1}^{m} x_i}{m^2\bar{x}^2} = \sigma^2\frac{\cancel{m\bar{x}}}{m^{\cancel{2}}\bar{x}^{\cancel{2}}}$$

$$= \frac{\sigma^2}{m\bar{x}} = \frac{\sigma^2}{\sum_{i=1}^{m} x_i}.$$

b. The ordinary least square is obtained as

$$\hat{\beta}_{OLS} = \underset{\beta \in \mathbb{R}}{\arg\min} \sum_{i=1}^{m}(y_i - x_i\beta)^2$$

$$\Longrightarrow \ell'(\beta) = -2\sum_{i=1}^{m}(y_i - x_i\beta)x_i \Longrightarrow \cancel{-2}\sum_{i=1}^{m} x_i y_i = \cancel{-2}\beta\sum_{i=1}^{m} x_i^2.$$

$$\Longrightarrow \hat{\beta}_{OLS} = \frac{\frac{1}{m}\sum_{i=1}^{m} x_i y_i}{\frac{1}{m}\sum_{i=1}^{m} x_i^2 = \bar{x}_2}.$$

$$\mathrm{var}(\hat{\beta}_{OLS}) = \mathrm{var}\left(\frac{\frac{1}{m}\sum_{i=1}^{m} x_i y_i}{\bar{x}_2}\right) = \frac{1}{m^2\bar{x}_2}\,\mathrm{var}\left(\sum_{i=1}^{m} x_i y_i\right) = \frac{1}{m^2\bar{x}_2}\sum_{i=1}^{m} x_i\,\mathrm{var}(y_i)$$

independence

$$= \sigma^2 \frac{\sum_{i=1}^m x_i^3}{m^2 \bar{x}_2} = \frac{\sigma^2}{m} \frac{\bar{x}_3}{\bar{x}_2^2}, \quad \text{where} \quad \bar{x}_3 = \frac{1}{m}\sum_{i=1}^m x_i^3 \, ; \; \bar{x}_2 = \frac{1}{m}\sum_{i=1}^m x_i^2 .$$

c. We want to show that $var(\hat{\beta}_{wls}) \leq var(\hat{\beta}_{ols})$ that is

$$\cancel{\frac{\sigma^2}{m}} \cdot \frac{1}{\bar{x}} \leq \cancel{\frac{\sigma^2}{m}} \frac{\bar{x}_3}{\bar{x}_2^2} \iff \frac{1}{\bar{x}} \leq \frac{\bar{x}_3}{\bar{x}_2^2} \iff \boxed{\bar{x}_3 \bar{x} \geq \bar{x}_2^2}$$

Set

$$y_i = x_i^{3/2}, \quad z_i = x_i^{1/2}, \qquad i = 1, \dots, m.$$

Consequently, using Cauchy-Schwartz

$$\left(\frac{1}{m}\sum_{i=1}^m y_i z_i\right)^2 \leq \left(\frac{1}{m}\sum_{i=1}^m y_i^2\right)\left(\frac{1}{m}\sum_{i=1}^m z_i^2\right)$$

Substituting:

$$\left(\frac{1}{m}\sum_{i=1}^m x_i^{3/2} x_i^{1/2}\right)^2 \leq \left(\frac{1}{m}\sum_{i=1}^m x_i^{\frac{3}{2}\cdot 2}\right)\left(\frac{1}{m}\sum_{i=1}^m x_i^{\frac{1}{2}\cdot 2}\right)$$

$$\Downarrow$$

$$\left(\frac{1}{m}\sum_{i=1}^m x_i^2\right)^2 \leq \left(\frac{1}{m}\sum_{i=1}^m x_i^3\right)\left(\frac{1}{m}\sum_{i=1}^m x_i\right)$$

$$\Downarrow$$

$$\boxed{\bar{x}_2^2 \leq \bar{x}_3 \bar{x}.}$$

Which implies that $var(\hat{\beta}_{wls}) \leq var(\hat{\beta}_{ols})$.