Exercise B

Let y be a random variable with negative binomial distribution, with $\tilde{\pi} \in (0,1)$ and $y \in S = \{k, k+1, \dots\}$, so that

$$p(y;\tilde{\pi}) = \binom{y-1}{k-1} \tilde{\pi}^{k} (1-\tilde{\pi})^{y-k}, \qquad y = k, k+1, \dots$$

Thus,

$$p(y;\tilde{\pi}) = \left(\frac{\tilde{\pi}}{1-\tilde{\pi}}\right)^{k} (1-\tilde{\pi})^{y} \binom{y-1}{k-1}$$

$$= \exp\left\{ y \underbrace{\log(1-\tilde{\pi})}_{=\theta} + k \log\left(\frac{\tilde{\pi}}{1-\tilde{\pi}}\right) + \underbrace{\log\binom{y-1}{k-1}}_{=c(y)} \right\}$$

Hence

$$\boxed{\theta = \log(1-\tilde{\pi})} \implies 1-\tilde{\pi} = e^{\theta} \implies \tilde{\pi} = 1 - e^{\theta}$$

$$\implies k \log\left(\frac{\tilde{\pi}}{1-\tilde{\pi}}\right) = k \log\left(\frac{1-e^{\theta}}{e^{\theta}}\right) = -\log\left(\frac{e^{\theta}}{1-e^{\theta}}\right)$$

Summarising, we obtain

$$\theta = \log(1-\tilde{\pi}), \quad a(\phi) = 1, \quad b(\theta) = \log\left(\frac{e^{\theta}}{1-e^{\theta}}\right), \quad c(y) = \log\binom{y-1}{k-1}.$$

Mean function

$$b'(\theta) = \frac{\partial}{\partial\theta} k \log\left(\frac{e^{\theta}}{1-e^{\theta}}\right) = k \frac{\partial}{\partial\theta}\theta + \frac{\partial}{\partial\theta} - k \log(1-e^{\theta}) = k + \frac{k e^{\theta}}{1-e^{\theta}}$$

$$= k \frac{1-e^{\theta}+e^{\theta}}{1-e^{\theta}} = \frac{k}{1-e^{\theta}}$$

$$= \frac{k}{\tilde{\pi}} = \mu.$$

$$\implies \mu(\theta) = \frac{k}{1-e^{\theta}} = \frac{k}{\tilde{\pi}}.$$

Moreover, we get

$$1-e^{\theta} = \frac{\kappa}{\mu} \iff e^{\theta} = 1-\frac{\kappa}{\mu} = \frac{\mu-\kappa}{\mu} \implies \theta(\mu) = \log\left(\frac{\mu-\kappa}{\mu}\right) \quad \text{(Canonical link)}$$

<u>Variance function</u>

$$b''(\theta) = \frac{\partial}{\partial\theta}\frac{\kappa}{1-e^{\theta}} = \frac{\partial}{\partial\theta}\kappa\left(1-e^{\theta}\right)^{-1} = \kappa\left[-\left(1-e^{\theta}\right)^{-2}\right]\left(-e^{\theta}\right)$$

$$= \kappa\frac{e^{\theta}}{\left(1-e^{\theta}\right)^{2}} = \kappa\frac{1-\tilde{\pi}}{\tilde{\pi}^{2}} \quad \text{(See above)}$$

$$\implies \operatorname{var}(y) = \underset{\overset{\shortparallel}{1}}{a(\phi)}\, b''(\theta) = \kappa\frac{(1-\tilde{\pi})}{\tilde{\pi}^{2}}.$$

Moreover

$$\operatorname{var}(y) = \underset{\overset{\shortparallel}{\mu}}{\frac{\kappa}{\tilde{\pi}}}\cdot\frac{\kappa}{\tilde{\pi}}\frac{1}{\kappa}(1-\tilde{\pi}) = \left(\frac{\kappa}{\tilde{\pi}}\right)^{2}\left(\frac{1}{\kappa}-\frac{\tilde{\pi}}{\kappa}\right) = \mu^{2}\left(\frac{1}{\kappa}-\frac{1}{\mu}\right) = \frac{\mu^{2}}{\kappa}-\mu$$

$$= V(\mu).$$

If κ were unknown, then this is <u>not</u> anymore an exponential family.