Exercise E

Setup: $\mathbb{E}[y_i] = \mu_i = g^{-1}(\beta x_i)$, with $p = 1$.

Case $y_i \sim ED(\mu_i, \phi)$. This is a Gaussian distribution, with $y_i \sim N(\beta x_i, \phi)$.

known.

Canonical link is the identity

$$e(\beta) = -\frac{1}{2\phi}\sum_{i=1}^{m}(y_i - \beta x_i)^2$$

$$e^*(\beta) = \frac{\partial}{\partial\beta}e(\beta) = \frac{1}{\phi}\sum_{i=1}^{m}x_i(y_i - \beta x_i) \Longrightarrow \hat{\beta} = \frac{\sum_{i=1}^{m}x_i y_i}{\sum_{i=1}^{m}x_i^2}.$$

$$I(\beta) = \text{var}(e^*(\beta)) = \frac{1}{\phi^2}\sum_{i=1}^{m}x_i^2\,\text{var}(\underbrace{y_i - \beta x_i}_{=\phi}) = \frac{\cancel{\phi}}{\phi^{\cancel{2}}}\sum_{i=1}^{m}x_i^2 = \frac{m}{\phi}\bar{x}_2.$$

$$= -e^{**}(\beta)$$

They coincide

Because of the properties of the Gaussian distribution,

$$\hat{\beta} = \frac{1}{m\bar{x}_2}\cdot\sum_{i=1}^{m}x_i y_i$$ is a Gaussian (linear combination of Gaussians), with

mean and variance

$$\mathbb{E}[\hat{\beta}] = \frac{1}{m\bar{x}_2}\sum_{i=1}^{m}x_i\,\underbrace{\mathbb{E}[y_i]}_{=\beta x_i} = \beta\,\frac{m\bar{x}_2}{m\bar{x}_2} = \beta$$

$$\text{var}(\hat{\beta}) = \frac{1}{m^2\bar{x}_2^2}\sum_{i=1}^{m}x_i^2\,\text{var}(y_i) = \phi\,\frac{\cancel{m\bar{x}_2}}{m^{\cancel{2}}\bar{x}_2^{\cancel{2}}} = \frac{\phi}{m\bar{x}_2} = I^{-1}$$

$$\Longrightarrow \hat{\beta} \sim N\left(\beta, \frac{\phi}{m\bar{x}_2}\right), \text{ and } \hat{\mu}_i = \hat{\beta}x_i \sim N\left(\beta x_i, \frac{\phi x_i^2}{m\bar{x}_2}\right)$$

C.I. for μ_i is $\hat{\mu}_i \pm z_{1-\alpha/2}\sqrt{\frac{\phi x_i^2}{m\bar{x}_2}}$

this is known, so the interval is exact.

Case $y_i \sim ED(\mu_i, \mu_i)$. This is a Poisson, so that $y_i \sim \text{Poisson}(\mu_i)$, $\mu_i = e^{\beta x_i} = \alpha^{x_i}$.

$$\ell(\beta) = \sum_{i=1}^{m} y_i \beta x_i - e^{\beta x_i}$$

$$\ell^*(\beta) = \sum_{i=1}^{m} x_i y_i - x_i e^{\beta x_i} \implies \hat{\beta} \text{ solves } \sum_{i=1}^{m} x_i y_i = \sum_{i=1}^{m} x_i e^{\hat{\beta} x_i}$$

$$I(\beta) = \text{var}(\ell^*(\beta)) = \sum_{i=1}^{m} x_i^2 \text{var}(y_i) = \sum_{i=1}^{m} x_i^2 e^{\beta x_i} = X^T W X.$$

$$\hat{\beta} \sim N\left(\beta, \left(\sum_{i=1}^{m} x_i^2 e^{\beta x_i}\right)^{-1}\right)$$

asymptotically

$$\implies \hat{\eta}_i = \hat{\beta} x_i \sim N\left(\beta x_i, \frac{x_i^2}{\sum_{i=1}^{m} x_i^2 e^{\beta x_i}}\right)$$ therefore the confidence interval is

$$CI = \hat{\eta}_i \pm z_{1-\alpha/2} \sqrt{\frac{x_i^2}{\sum_{i=1}^{m} x_i^2 e^{\beta x_i}}}.$$ (Approximate CI)

The CI for μ_i can be simply obtained by exponentiating the above terms, obtaining

$$\exp\left(\hat{\eta}_i \pm z_{1-\alpha/2} \sqrt{\frac{x_i^2}{\sum_{i=1}^{m} x_i^2 e^{\beta x_i}}}\right).$$ (Approximate CI).

This is <u>not</u> Wald.

Case $y_i \sim ED(\mu_i, \mu_i(1-\mu_i))$. This is a binary variable $y_i \sim \text{Ber}(\mu_i)$, with $\text{logit}(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) = \beta x_i$.

$$\ell(\beta) = \sum_{i=1}^{m} y_i \beta x_i - \log(1 + e^{\beta x_i})$$

$$\ell^*(\beta) = \sum_{i=1}^{m} x_i y_i - x_i \frac{e^{\beta x_i}}{1 + e^{\beta x_i}} \implies \hat{\beta} \text{ solves } \sum_{i=1}^{m} x_i y_i = \sum_{i=1}^{m} x_i \frac{e^{\hat{\beta} x_i}}{1 + e^{\hat{\beta} x_i}}.$$

$$I(\beta) = \sum_{i=1}^{m} x_i^2 \frac{e^{\beta x_i}}{(1 + e^{\beta x_i})^2} = X^T W X$$

$$\hat{\beta} \sim N\left(\beta, \underbrace{\left(\sum_{i=1}^{m} x_i^2 \frac{e^{\beta x_i}}{(1+e^{\beta x_i})^2}\right)^{-1}}_{= var(\beta)}\right) \Rightarrow \hat{\eta}_i \sim N(x_i\beta, x_i^2 var(\beta))$$

asymptotically

$$CI = \hat{\eta}_i \pm z_{1-\alpha/2}\sqrt{x_i^2 var(\hat{\beta})}.$$ (Approximate CI)

The CI for p_i can be simply obtained by transforming the above terms, obtainin

$$\frac{e^{\hat{\eta}_i \pm z_{1-\alpha/2}\sqrt{x_i^2 var(\hat{\beta})}}}{1 + e^{\hat{\eta}_i \pm z_{1-\alpha/2}\sqrt{x_i^2 var(\hat{\beta})}}}.$$ (Approximate CI).

This is <u>not</u> Wald.