Exercise A

Let $y \sim \text{Binomial}(m, \pi)$. Consider the logit parametrization $\theta = \text{logit}\left(\frac{\pi}{1-\pi}\right)$. The corresponding Wald test is based on the statistic

$$z = \frac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})}, \quad \text{where } \hat{\theta} = \text{logit}\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right), \text{ and } \hat{\pi} = \frac{y}{m}.$$

→ Equivariance property of the MLE

→ MLE for a Binomial proportion.

Specializing well-known results of GLM (or applying the delta method), we get

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{1}{m\hat{\pi}(1-\hat{\pi})}}$$

Such a test reject for high values of z. However, if $\theta_0 = 0$, note that when

$m = 25$, $y = 23$

$$\hat{\theta} = 2.44; \quad \hat{se}(\hat{\theta}) = 0.7372; \quad z = 3.31$$

Moreover, if $m = 25$, $y = 24$, then

$$\hat{\theta} = 3.178; \quad \hat{se}(\hat{\theta}) = 1.021; \quad z = 3.11$$

This is not what we would expect. Larger values of $\hat{\theta}$ should provide more evidence against the null hypothesis, not less. The log-likelihood ratio test does not have this problem, in fact the test is

$$r_0 = \text{sign}(\hat{\theta} - \theta_0)\sqrt{2\left[\ell(\hat{\theta}) - \ell(\theta_0)\right]}$$

and

$$\ell(\hat{\theta}) - \underbrace{\ell(0)}_{= c} = y \log\left(\frac{y}{m}\right) + (m-y)\log\left(\frac{m-y}{m}\right) - c$$

is a monotone increasing function of y.

(Test are $r_0 = 4.461$ when $y = 23$, and $r_0 = 5.12$ when $y = 24$).