Exercise D

The first part of the exercise is already described in the slides.

$$\hat{\beta}_{(m+1)} = (X^T_{(m+1)} X_{(m+1)})^{-1} X^T_{(m+1)} y_{(m+1)} = V_{(m+1)} X^T_{(m+1)} y_{(m+1)}$$

$$V_{(m+1)} = V_{(m)} - v_m V_{(m)} x_{m+1} x^T_{m+1} V_{(m)}, \quad v_m = \frac{1}{1 + x^T_{(m+1)} V_{(m)} x_{(m+1)}}$$

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$$\hat\beta_{(m+1)} = V_{(m+1)}\left[X^T_{(m)} y_{(m)} + x_{m+1}\, y_{m+1}\right]$$

$$= \left[V_{(m)} - v_m V_{(m)} x_{m+1} x^T_{m+1} V_{(m)}\right]\left[X^T_{(m)} y_{(m)} + x_{m+1}\, y_{m+1}\right]$$

$$= \underbrace{V_{(m)} X^T_{(m)} y_{(m)}}_{\hat\beta_{(m)}} + V_{(m)} x_{m+1} y_{m+1} - v_m V_{(m)} x_{m+1} x^T_{m+1} \underbrace{V_{(m)} X^T_{(m)} y_{(m)}}_{\hat\beta_{(m)}} +$$

$$- v_m V_{(m)} x_{m+1} x^T_{m+1} V_{(m)} x_{m+1}\, y_{m+1}$$

$$= \hat\beta_{(m)} + v_m V_{(m)} x_{m+1}\left[\frac{1}{v_m} y_{m+1} - x^T_{m+1}\hat\beta_{(m)} - x^T_{m+1} V_{(m)} x_{m+1} y_{m+1}\right]$$

$$= \hat\beta_{(m)} + v_m V_{(m)} x_{m+1}\left[y_{m+1}\underbrace{\left(\frac{1}{v_m} - x^T_{m+1} V_{(m)} x_{m+1}\right)}_{=1} - x^T_{m+1}\hat\beta_{(m)}\right]$$

$$1 + x^T_{(m+1)} V_{(m)} x_{(m+1)} - x^T_{(m+1)} V_{(m)} x_{(m+1)}$$

$$= \hat\beta_{(m)} + \underbrace{v_m V_{(m)} x_{m+1}}_{K_m}\left(y_{m+1} - x^T_{m+1}\hat\beta_{(m)}\right)$$

For the second part, we need to show that

$$\|y_{(m+1)} - X_{(m+1)}\hat\beta_{(m+1)}\|^2 = \|y_{(m)} - X_{(m)}\hat\beta_{(m)}\|^2 + v_{(m)}\, e^2_{m+1}.$$

Note in the first place that

$$\|y_{(m+1)} - X_{(m+1)}\hat\beta_{(m+1)}\|^2 = y^T_{(m+1)}\left[I_{m+1} - H_{(m+1)}\right] y_{(m+1)} =$$

$$= y^T_{(m+1)}\left[I - X_{(m+1)} V_{(m+1)} X^T_{(m+1)}\right] y_{(m+1)} = y^T_{(m+1)}\left[y_{(m+1)} - X_{(m+1)}\hat\beta_{(m+1)}\right]$$

Now, we can "seperate" the additional term.

$$= \begin{bmatrix} y_{(m)} \\ y_{m+1} \end{bmatrix}^T \left\{ \begin{bmatrix} y_{(m)} \\ y_{m+1} \end{bmatrix} - \begin{bmatrix} X_{(m)} \\ x^T_{m+1} \end{bmatrix} \hat{\beta}_{(m+1)} \right\} =$$

$$= y_{(m)}^T \left(y_{(m)} - X_{(m)} \hat{\beta}_{(m+1)} \right) + y_{m+1} \left(y_{m+1} - x^T_{m+1} \hat{\beta}_{m+1} \right) =$$

$$= y_{(m)}^T \left(y_{(m)} - X_{(m)} \hat{\beta}_{(m)} - X_{(m)} K_m e_{m+1} \right) + y_{m+1} \Big(\underbrace{y_{m+1} - x^T_{m+1} \hat{\beta}_m}_{e_{m+1}} - x^T_{m+1} K_m e_{m+1} \Big)$$

$$= \underbrace{y_{(m)}^T \left(y_{(m)} - X_{(m)} \hat{\beta}_{(m)} \right)}_{= \| y_{(m)} - X_{(m)} \beta_{(m)} \|^2} - y_{(m)}^T X_{(m)} K_m e_{m+1} + y_{m+1} \left(e_{m+1} - x^T_{m+1} K_m e_{m+1} \right)$$

$$= \| y_{(m)} - X_{(m)} \hat{\beta}_{(m)} \|^2 + e_{m+1} \Big[y_{m+1} - y_{m+1} x^T_{m+1} K_m - \underbrace{y_{(m)}^T X_{(m)}}_{= \hat{\beta}_{(m)}^T V_{(m)}^{-1}} \overset{v_m V_{(m)} x_{m+1}}{K_m} \Big]$$

$$= \| y_{(m)} - X_{(m)} \hat{\beta}_{(m)} \|^2 + e_{m+1} \Big[\underbrace{y_{m+1} - y_{m+1} x^T_{m+1} K_m}_{= v_m y_{m+1}} - \hat{\beta}_{(m)}^T v_m x_{m+1} \Big]$$

$$= y_{m+1} \left(1 - x_{m+1} v_m V_{(m)} x_{m+1} \right)$$

$$= y_{m+1} \left(1 - \frac{x_{m+1} V_{(m)} x_{m+1}}{1 + x_{m+1} V_{(m)} x_{m+1}} \right) = y_{m+1} \frac{1}{1 + x_{m+1} V_{(m)} x_{m+1}}$$

$$= \| y_{(m)} - X_{(m)} \hat{\beta}_{(m)} \|^2 + e_{m+1} \left(v_m y_{m+1} - v_m x^T_{m+1} \hat{\beta}_{(m)} \right)$$

$$= \| y_{(m)} - X_{(m)} \hat{\beta}_{(m)} \|^2 + v_m e^2_{m+1}.$$