Exercise F

Let us consider a linear tranformation of the covariates, that is

$$Z = XA,$$

where A is a $p \times p$ invertible matrix. Then, let us consider the models

$$y = Z\gamma + \varepsilon$$
$$y = X\beta + \varepsilon$$

producing the least squares estimates

$$\hat{\gamma} = (Z^T Z)^{-1} Z^T y; \qquad \hat{\beta} = (X^T X)^{-1} X^T y.$$

Note that:

$$\hat{\gamma} = (Z^T Z)^{-1} Z^T y = \left[(XA)^T (XA)\right]^{-1} (XA)^T y = \left[A^T X^T X A\right]^{-1} A^T X^T y$$

$$= A^{-1} (X^T X)^{-1} \cancel{(A^T)^{-1}} \cancel{(A^T)} X^T y =$$

$$= A^{-1} (X^T X)^{-1} X^T y$$

$$\Rightarrow \hat{\gamma} = A^{-1} \hat{\beta}$$

Consequently, we have that

$$\boxed{\hat{y} = Z\hat{\gamma} = XA \cdot A^{-1} \hat{\beta} = X\hat{\beta}}$$

Standardization is just a special case (assume data have been centered)

$$z_{i1} = x_{i1} = 1; \qquad z_{ij} = \frac{x_{ij} - \bar{x}_j}{s_j} = a_j x_{ij} + b_j; \qquad j = 2, \dots, p, \text{ with } a_j = \frac{1}{s_j}, \; b_j = -\frac{\bar{x}_j}{s_j}$$

$$Z = XA, \text{ with } A = \begin{bmatrix} 1 & b_2 & \cdots & b_p \\ 0 & a_2 & & \\ \vdots & & \ddots & \emptyset \\ 0 & \emptyset & & a_p \end{bmatrix}$$