Exercise B

Let us consider the inequality $B \preccurlyeq A$, between two squared matrices A and B of dim $p \times p$.
The symbol $\preccurlyeq$ means that $A - B$ is positive semi-definite

Assume (x_i, y_i) is linear, therefore $y = X\beta + \varepsilon$, with $\varepsilon_i \overset{iid}{\sim} \mathcal{F}$, with $\mathbb{E}[\varepsilon_i] = 0$, and $var(\varepsilon_i) = \sigma^2$.

Let $\hat{\beta}_{ols} = (X^T X)^{-1} X^T y$ and let $\tilde{\beta} = Ay$ such that $\tilde{\beta} \neq \beta_{ols}$ and $\mathbb{E}[\tilde{\beta}] = \beta$. Then

Decompose

$$\text{Let } C = A - (X^T X)^{-1} X^T y \Rightarrow A = (X^T X)^{-1} X^T y + C.$$

By assumption, we have that $\mathbb{E}[\tilde{\beta}] = \beta$ and

$$\mathbb{E}[\tilde{\beta}] = \mathbb{E}[Ay] = AX\beta = \cancel{(X^T X)^{-1} X^T X}\beta + CX\beta = \beta(I + CX).$$

Hence, we <u>must</u> have that $\boxed{CX = 0}$. Now, note that

$$\tilde{V} = var(Ay) = \sigma^2 AA^T =$$

$$= \sigma^2 [(X^T X)^{-1} X^T + B][(X^T X)^{-1} X^T + B]^T$$

$$= \sigma^2 [(X^T X)^{-1} X^T + B][X(X^T X)^{-1} + B^T]$$

$$= \sigma^2 [\cancel{(X^T X)^{-1} X^T X}(X^T X)^{-1} + \cancel{(X^T X)^{-1} \underbrace{X^T B^T}_{\substack{(BX)^T \\ = 0}}} + \cancel{\underbrace{BX}_{=0} (X^T X)^{-1}} + BB^T]$$

$$= \sigma^2 (X^T X)^{-1} + \sigma^2 BB^T$$

$$= V + \sigma^2 BB^T$$

$\Rightarrow \tilde{V} - V = \sigma^2 BB^T$, which is positive semi-definite (same proof of the matrix $(X^T X)$).